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Apollonian arrangements of spheres in *d*-dimensional space

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Abstract. The problem of arranging d-dimensional spheres in such a way that each sphere touches exactly $d+1$ others is formulated by means of linear recursions involving $d+1$ different $(d+2)\times(d+2)$ matrices. The average curvatures are calculated exactly for any dimension. Special scaling limits for the powers of one matrix or of the products of two matrices show interesting behaviour: the corresponding curvatures either diverge quadratically, or form a six-step cycle (only for $d=3$ and one matrix), or show ergodic behaviour, sometimes with universal invariant measure. The latter type of behaviour is also found if the results are formally extended to non-integer values of d.

1. Introduction

Recently, the problem of the Apollonian tiling (after Apollonius of Perge, *c*. 200 BC) of a plane by mutually touching circles has received considerable attention. Mathematically, this is an example of a *packing* of spheres, which is defined as a set of non-intersecting spheres all contained completely within a bounded region of space. Such a packing is called *complete* if the total volume of these spheres (of which there must then be infinitely many) equals the volume of the container. A packing is called *osculatory* or *Apollonian* if the spheres are tangent to each other. It is known, that in two [1] and three [2] dimensions, osculatory packings are complete, whereas their analogues in more than three dimensions contain intersecting spheres. These *Apollonian arrangements* (with packings as special cases) form the subject of the present paper.

Non-osculatory, complete packings are also of great interest; these have recently been studied extensively in two dimensions [3, 4], and also in dimensions larger than three to provide alternatives to the Apollonian arrangements, which are not packings [5]. Osculatory packings in two dimensions with extra symmetry properties are space-filling bearings [6, 7]. All of these constructions possess a group-theoretical backgrounds [8, 9] (see also [4]).

The (multi) fractal nature of the set remaining after removal of all open spheres was recognized by Mandelbrot [10] for the two-dimensional case (Apollonian gasket). In general, some care is necessary to identify the packing constant of a packing with a fractal dimension [11] (see also [12]). Several results for the fractal dimension of the Apollonian gasket have been obtained [13–17]. (For the other packings listed above, see the references given.) It has been suggested, that Apollonian packings may provide models for foams, turbulence, liquid crystal textures and tectonic fault line movements.

The present article is a step towards a generalization of Apollonian packing to dimensions higher than two. In d dimensions, $d+2$ spheres can simultaneously touch each other. The relation between the curvatures of these spheres were first given by Soddy [18] for $d=2$ and $d=3$, and by Gosset [19] for the general case. A simple modern derivation can be found in [9]; see also section 2. For $d > 2$, the situation is quite different from the $d = 2$ case, since the $d + 2$ touching spheres have interstices, which are multiply connected to the space far from the spheres. Therefore, a recursive sequence of spheres will, in general, contain some with positive curvature ('inside') and some with negative curvature, which envelop the original ones. Also, spheres from sufficiently different recursive depths can be identical (this happens only for $d = 3$) or they can intersect each other (for $d > 3$) [2].

This article is organized as follows: in section 2, the recursion relations in d dimensions are formulated in terms of $d + 1$ matrices, which are the analogues of the Boyd matrices [15]. From these, the average curvature after n iterations is found to increase exponentially for all d. In section 3, the scaling limit for recursion using only powers of one of the matrices is studied. The results are: (i) for $d = 2$, the curvatures diverge quadratically, as already shown in [9]; for $d = 3$, the curvatures form a six-step cycle, which defines a peculiar arrangement of 18 spheres; (iii) for $d > 3$, the curvatures fill up the whole interval of allowed values with distribution function

$$\rho(x) = [\pi(1 - x^2)^{1/2}]^{-1} \tag{1.1}$$

if the allowed interval is mapped linearly on to $[-1, 1]$. In section 4 the scaling limit for powers of a product of two matrices is studied. Here the results are: (i) *exponential* divergence of the curvatures for $d = 2$ [9]; (ii) quadratic divergence for $d = 3$; and (iii) very complex behaviour for $d > 3$: depending on the initial arrangement of spheres, the whole allowed interval or only part of it is filled up; in some cases, the distribution function seems to be the sum of two contributions of the type equation (1.1). In the final section, the results are briefly discussed.

2. Matrix formulation and the average curvature

Let the curvatures of $d + 1$ mutually touching spheres in d dimensions be given by $a_1, a_2, \ldots, a_{d+1}$. Then the $(d + 2)$th sphere touching all $d + 1$ others has a curvature s given by [9, 18, 19]

$$d\left[s^2 + \sum_{i=1}^{d+1} a_i^2\right] = \left[s + \sum_{i=1}^{d+1} a_i\right]^2. \tag{2.1}$$

If s_j is the curvature of the sphere touching the 'central' one with curvature s and all of the others with the exception of the one with curvature a_j, then a_j and s_j necessarily are the two solutions of the same quadratic equation:

$$d\left[s^2 + x^2 + \sum_{i\neq j} a_i^2\right] = \left[s + x + \sum_{i\neq j} a_i\right]^2 \tag{2.2}$$

so that one has

$$s_j = [2/(d-1)]\left(s + \sum_{i\neq j} a_i\right) - a_j. \tag{2.3}$$

Therefore, if the two sets of $d + 2$ mutually touching spheres are represented by $(d + 2)$-dimensional vectors as

$$v^T = (a_1, a_2, \ldots, a_{d+1}, s) \qquad \text{and} \qquad v_j^T = (a_1, \ldots, a_{j-1}, s, \ldots, a_{d+1}, s_j) \tag{2.4}$$

respectively, then the matrix M_j which maps $\boldsymbol{v}$ on $\boldsymbol{v}_j$ can be written as

$$M_j = BT(j, d+2) \qquad j = 1, \ldots, d+1 \tag{2.5}$$

where $T(j, d+2)$ is the matrix representative of the transposition $(j, d+2)$ and B is a fixed matrix with structure:

$$B = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & 1 & \\ g & g & \cdot\ \cdot & g & -1 \end{pmatrix} \qquad g = 2/(d-1). \tag{2.6}$$

All elements of B not on the diagonal or in the last row are zero. We define the Apollonian arrangement of spheres with seed $\boldsymbol{v}_0$ as the result of the application of all possible products of M-matrices on $\boldsymbol{v}_0$. This Apollonian arrangement of spheres contains at most $(d+1)^n$ different ones in the nth generation (due to n-fold matrix products).

The average curvature $\overline{a}_n$ in the limit of large n can be calculated by considering the $(d+2)$th element of the vector $\boldsymbol{w}_n$ obtained by applying the sum of all M_j-matrices to $\boldsymbol{v}_0$:

$$\boldsymbol{w}_n = \left[\sum_{j=1}^{d+1} M_j\right]^n \boldsymbol{v}_0 \equiv N^n \boldsymbol{v}_0 \tag{2.7a}$$

$$\overline{a}_n = (d+1)^{-n} \boldsymbol{w}_n(d+2). \tag{2.7b}$$

The matrix N has the form:

$$N = \begin{pmatrix} d & & & & 1 \\ & d & & & 1 \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & d & 1 \\ h & h & \cdot\ \cdot & h & 2h \end{pmatrix} \qquad h = (d+1)/(d-1) \tag{2.8}$$

with all elements not on the diagonal or in the last row or column equal to zero. The matrix N has d eigenvalues equal to d corresponding to the eigenspace of vectors of the form

$$(b_1, b_2, \ldots, b_{d+1}, 0)^T \qquad \text{with } \sum_{i=1}^{d+1} b_i = 0 \tag{2.9}$$

and two eigenvalues given as (+ sign for e_1)

$$e_{1,2} = \left\{d^2 + d + 2 \pm [d(d^3 - 2d^2 + 9d + 8)]^{1/2}\right\} / [2(d-1)] \tag{2.10}$$

with eigenvectors of the form

$$(1, 1, \ldots, 1, b)^T. \tag{2.11}$$

Therefore, $\overline{a}_n$ grows exponentially for large n as

$$\overline{a}_n \simeq \mu[e_1/(d+1)]^n \tag{2.12}$$

with μ depending on $\boldsymbol{v}_0$. It is to be noted, that equation (2.12) represents the average of all curvatures *with sign*, so that the occurrence of negative curvatures for $d > 2$ implies that the average *absolute* curvature grows faster than equation (2.12) in this case.

For $d = 2$, $e_1 = 4 + \sqrt{13}$; this implies a lower bound on the Hausdorff dimension d_f of the Apollonian gasket:

$$d_f \geqslant \ln(3)/[\ln(4+\sqrt{13}) - \ln(3)] \simeq 1.181. \tag{2.13}$$

This is not very good as an estimate for d_f (the best estimate is $d_f = 1.305\,686\,729(10)$ [17]), but constitutes a very simple proof that $d_f > 1$ holds.

3. Scaling limit for powers of one matrix

In this section, we study the curvatures of spheres obtained from an initial vector of the form

$$v_0 = (1, 1, \ldots, 1, s) \qquad s = \left\{d + 1 + [2d(d+1)]^{1/2}\right\}/(d-1) \tag{3.1}$$

by applying the powers of one M_j-matrix to it. We choose $j = d + 1$ here. Then if b_n is the $(d+2)$th element of $(M_{d+1})^n v_0$, one has the recursion relation

$$b_n = 2(b_{n-1} + d)/(d-1) - b_{n-2} \qquad b_0 = 1, b_1 = s. \tag{3.2}$$

In the case $d = 2$, equation (3.2) is easily solved to give [9]

$$b_n = 1 + (s-1)n + 2n(n-1). \tag{3.3}$$

For $d = 3$, equation (3.2) gives rise to a six-step cycle:

$$b_{6n} = 1 \qquad b_{6n+1} = 2 + \sqrt{6} \qquad b_{6n+2} = 4 + \sqrt{6}$$
$$b_{6n+3} = 5 \qquad b_{6n+4} = 4 - \sqrt{6} \qquad b_{6n+5} = 2 - \sqrt{6}.$$

This (together with the results for other values of j) describes a beautiful arrangement of 18 spheres: four touching spheres with curvature 1 form a tetrahedron; the inscribed sphere has curvature $2+\sqrt{6}$ and the sphere enclosing them has curvature $\sqrt{6}-2$ (which is counted as negative, since it encloses all others). In each of the four interstices between three of the radius-1 spheres, there fit exactly three spheres with curvatures (from inwards out) $4+\sqrt{6}$, 5 and $4-\sqrt{6}$.

For all cases with $d > 3$, numerical solution of (3.2) shows that every value in the interval

$$[b_{\min}, b_{\max}], b_{\max,\min} = [d \pm \{2d(d-1)\}^{1/2}]/(d-2) \tag{3.4}$$

occurs as a possible curvature value. Equation (3.4) gives the interval for the possible curvatures b in vectors of the form

$$(1, 1, \ldots, 1, a, b)^T. \tag{3.5}$$

If the interval of (3.4) is mapped linearly on to $[-1, 1]$, then the distribution function describing the density in this interval is numerically found to be given by (1.1) (shown as figure 1(a)) *universally* for all $d > 3$. (Checked up to $d = 20$ and for selected larger values of d.) This ergodic behaviour with invariant measure (1.1) is also found for all non-integral values of d that have been tested (equation (3.2) is well defined for all real d). In all cases, the results do not change if the initial vector v_0 is replaced by another of the form of equation (3.5). The distribution of equation (1.1) is well known to occur frequently for quadratic maps of the interval on itself [20] as well as in other contexts. It is not clear why it should show up in the present problem.

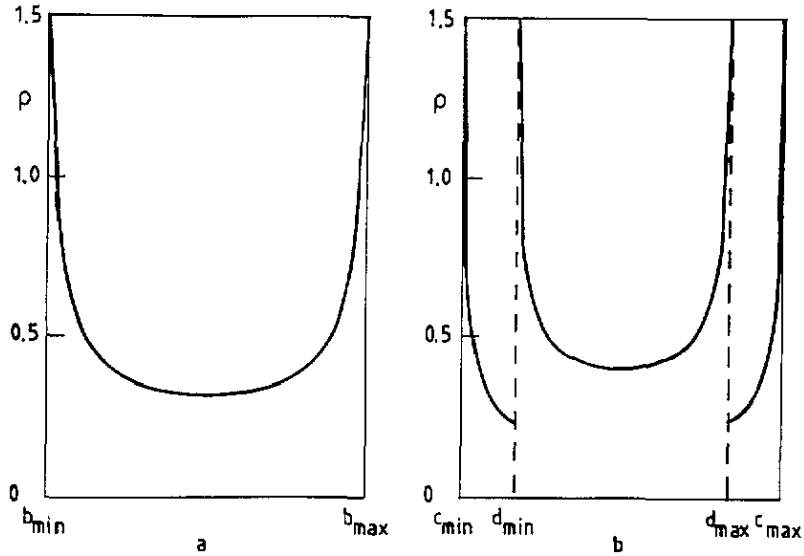

Figure 1. (a) The universal density function of equation (1.1) for one-matrix iterations for $d > 3$. (b) The density function for two-matrix iterations for $d > 3$ and initial vector v_0 of equation (3.1). The values of $d_{\min}$ and $d_{\max}$ here are for the case $d = 4$, but the general picture is always similar.

4. Scaling limit for powers of a product of two matrices

In this section, the sequence of vectors

$$u_{2n} = (M_d M_{d+1})^n v_0 \qquad u_{2n-1} = M_{d+1}(M_d M_{d+1})^{n-1} v_0 \tag{4.1}$$

with v_0 still given by (3.1) is studied. By consideration of the possible curvatures occurring in vectors of the form

$$(1, 1, \ldots, 1, a, b, c)^T \tag{4.2}$$

one finds as a possible interval for the $(d+2)$th elements of u_n:

$$[c_{\min}, c_{\max}], c_{\max,\min} = \left\{d - 1 \pm [2(d-1)(d-2)]^{1/2}\right\} / (d-3) \tag{4.3}$$

which is the same as (3.4), but with d replaced by $d - 1$.

Equations (4.1) are equivalent to the following set of recursion relations for the last three elements a_n, b_n and c_n of u_n:

$$\begin{aligned}
a_{2n+1} &= a_{2n} \\
b_{2n+1} &= c_{2n} \\
c_{2n+1} &= 2 - b_{2n} + 2(a_{2n} + b_{2n})/(d-1) \\
a_{2n+2} &= c_{2n+1} \\
b_{2n+2} &= c_{2n} \\
c_{2n+2} &= \frac{2d+2}{d-1} - a_{2n}\frac{(d+1)(d-3)}{(d-1)^2} - b_{2n}\frac{2}{d-1} + c_{2n}\frac{2d+2}{(d-1)}2.
\end{aligned} \tag{4.4}$$

For the case $d = 2$, Söderberg [9] already proved the exponential divergence of the curvatures. For $d = 3$, there is a closed subsystem of equations (4.4), due to the presence of a factor $d - 3$ in the last equation:

$$b_{2n+2} = c_{2n} \qquad c_{2n+2} = 2c_{2n} - b_{2n} + 4 \qquad b_0 = 1 \qquad c_0 = s = 2 + \sqrt{6}. \tag{4.5}$$

This leads to the quadratically divergent solution

$$c_{2n-2} = 2n^2 + (\sqrt{6} - 1)n + 1. \tag{4.6}$$

This quadratic behaviour is then inherited by all coefficients. All curvatures are positive.

For $d > 3$, the starting value v_0 again leads to all values allowed by equation (4.3). Half of these values come from odd-numbered iterations and these seem to have a distribution of the form (1.1) again, but in an interval $[c_{\min}, d_{\max}]$ with $d_{\max}$ smaller than $c_{\max}$. Similarly, the even-numbered iterations fill up an interval $[d_{\min}, c_{\max}]$ with $d_{\min} > c_{\min}$. Numerically,

$$d_{\min} + d_{\max} = c_{\min} + c_{\max} = 2(d-1)/(d-3) \tag{4.7}$$

holds, so that the distribution function in $[c_{\min}, c_{\max}]$ is symmetric with four square root singularities (see figure 1(*b*) for the case $d = 4$). In table 1, the values for the boundaries of the sub-intervals are collected for $3 < d < 13$. The $d_{\max,\min}$ values are numerical results, of course.

Table 1. Intervals filled up by two-matrix powers with initial v_0.

Dimension	$c_{\min}$	$d_{\min}$	$d_{\max}$	$c_{\max}$
4	−0.464 10	0.688 68	5.311 31	6.464 10
5	−0.449 49	0.658 07	3.341 93	4.449 49
6	−0.441 52	0.637 16	2.696 17	3.774 85
7	−0.436 49	0.622 03	2.377 97	3.436 49
8	−0.433 03	0.610 56	2.189 43	3.233 03
9	−0.430 50	0.601 60	2.065 06	3.097 17
10	−0.428 57	0.594 40	1.977 03	3
11	−0.427 05	0.588 48	1.911 52	2.927 05
12	−0.425 82	0.583 54	1.860 90	2.870 27

The above picture for $d > 3$ becomes even more complicated, if instead of v_0, other initial vectors of the form of equation (4.2) are chosen. It transpires that the distribution always has four square root singularities placed symmetrically in $[c_{\min}, c_{\max}]$, but the covered part of the interval may be smaller or may even be concentrated on two non-overlapping intervals $[d_1, d_2]$ and $[d_3, d_4]$, which are embedded symmetrically in the allowed interval:

$$d_1 + d_4 = d_2 + d_3 \qquad d_1 - c_{\min} = c_{\max} - d_4. \tag{4.8}$$

Similar results are obtained if the recursions of equation (4.4) are formally extended to non-integer values of d.

5. Summary

The Apollonian recursion of spheres in d dimensions has been cast into matrix form, allowing for exact calculation of the average curvature after n generations for n large. The scaling limits for powers of one matrix and for powers of a product of two matrices have yielded exact results in $d = 2$ and $d = 3$ and numerical ones for $d > 3$ and for non-integer values of d. These are briefly listed here:

$d = 2$: Quadratic divergence for one matrix, exponential divergence for a matrix product.

$d = 3$: A fixed structure consisting of 18 spheres for one matrix, quadratic divergence for a matrix product.

$d > 3$ and d non-integer: Numerically, one matrix yields ergodic behaviour in the whole allowed interval independent of initial conditions. There is a universal invariant measure given by equation (1.1). A matrix product gives a distribution, which appears to be the sum of two terms of the type (1.1); the parts of the interval filled up depend on the initial conditions.

For the case $d > 3$, it remains to investigate these initial state dependencies in more detail. This is difficult because of the problem of visualizing the inital configuration and its iterates. For $d = 3$, the 18-sphere structure is further filled up in the manner described in [2].

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References

[1] Melzak Z A 1966 *Can. J. Math.* **18** 838
[2] Boyd D W 1973 *Can. J. Math.* **25** 303; 1973 *Math. Comput.* **27** 369
[3] Bessis D and Demko S 1990 *Commun. Math. Phys.* **134** 293
[4] Bullett S and Mantica G 1992 *Nonlinearity* **5** 1085
[5] Boyd D W 1974 *Pacific J. Math.* **50** 383
[6] Herrmann H J, Mantica G and Bessis D 1990 *Phys. Rev. Lett.* **65** 3223
[7] Manna S S and Vicsek T 1991 *J. Stat. Phys.* **64** 525
[8] Maxwell G 1982 *J. Algebra* **79** 78
[9] Söderberg B 1992 *Phys. Rev.* A **46** 1859
[10] Mandelbrot B B 1982 *The Fractal Geometry of Nature* (San Francisco, CA: Freeman)
[11] Sullivan D 1984 *Acta Math.* **153** 259
[12] Falconer K J 1986 *The Geometry of Fractal Sets* (Cambridge: Cambridge University Press)
[13] Hirst K E 1967 *J. London Math. Soc.* **42** 281
[14] Melzak Z A 1969 *Math. Comput.* **23** 169
[15] Boyd D W 1971 *Aeq. Math.* **7** 182; 1973 *Aeq Math.* **9** 99; 1973 *Mathematika* **20** 170
[16] Manna S S and Herrmann H J 1991 *J. Phys. A: Math. Gen.* **24** L481
[17] Thomas P B and Dhar D 1994 *J. Phys. A: Math. Gen.* **27** 2257
[18] Soddy T 1936 *Nature* **137** 1021
[19] Gosset T 1937 *Nature* **139** 62
[20] Collet P and Eckmann J-P 1980 *Iterated Maps on the Interval as Dynamical Systems* (Basel: Birkhäuser)